THE MATHEMATICS OF TASK DESIGN

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(subbing for Glenn Stevens and Al Cuoco)

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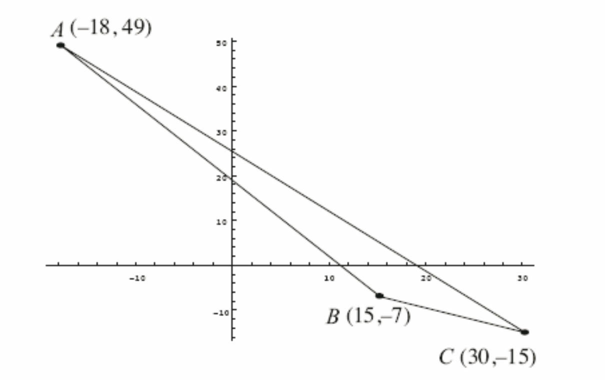
Outline

Outline

Getting Started

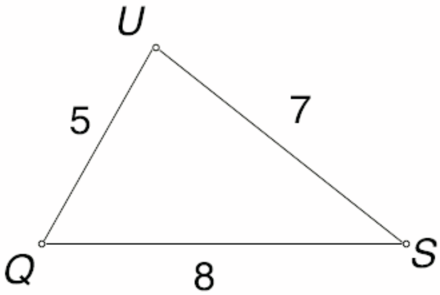
The vertices of a triangle have coordinates

$$(-18, 49), (15, -7), (30, -15)$$



A strange but nice triangle; how long are the sides?

GETTING STARTED



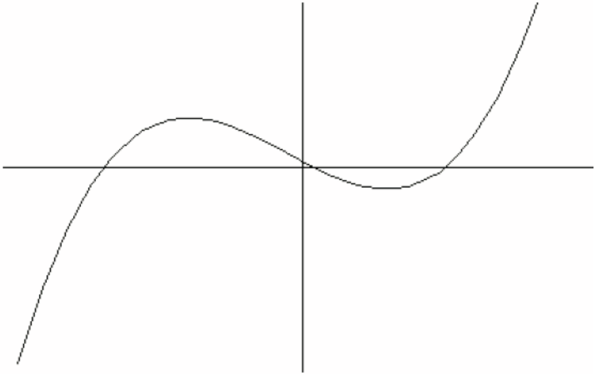
How big is angle Q?

GETTING STARTED

Fold up to make a box

What size cut-out maximizes the volume?

Getting Started

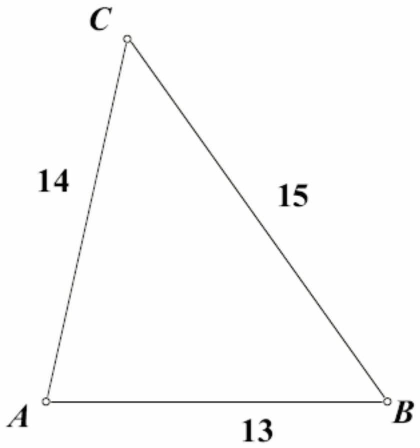


$f(x) = 140 - 144x + 3x^2 + x^3$

Find the zeros, extrema, and inflection points

Getting Started

Find the area of this triangle



Outline

An Algebraic Approach

How to Amaze Your Friends at Parties

A *Gaussian Integer* is a complex number of the form $a + bi$ where a and b are *integers*.

Pick your favorite Gaussian Integer (make $a > b$) and square it.

An Algebraic Approach

$$(a+bi)^2, N(a+bi)$$

$b \rightarrow$ $a \downarrow$	1	2	3
2	$3+4i, 5$		
3	$8+6i, 10$	$5+12i, 13$	
4	$15+8i, 17$	$12+16i, 20$	$7+24i, 25$
5	$24+10i, 26$	$21+20i, 29$	$16+30i, 34$
6	$35+12i, 37$	$32+24i, 40$	$27+36i, 45$
7	$48+14i, 50$	$45+28i, 53$	$40+42i, 58$
8	$63+16i, 65$	$60+32i, 68$	$55+48i, 73$

Why Does This Work?

- The norm of z, $N(z)$, is the product of z and its complex conjugate: $N(z) = z\,\overline{z}$.
- The following properties of norm hold:

1. $N(a+bi) = a^2 + b^2$, a non-negative integer.
2. $N(zw) = N(z)N(w)$ for all Gaussian integers z and w.
3. So

$$N(z^2) = (N(z))^2$$

Ah. . .

The Mathematics of Task Design

In general, if $z = a + bi$, then $N(z) = a^2 + b^2$, and

$$z^2 = (a^2 - b^2) + 2abi$$

so

$$(a^2 + b^2)^2 = (N(z))^2 = N\left(z^2\right) = (a^2 - b^2)^2 + (2ab)^2 \quad (*)$$

or (the famous)

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2 \quad (**)$$

Note: Identity $(**)$ holds in any commutative ring, including $\mathbb{Z}[i]$. And that's the key to making up lattice-point triangles with integer side-lengths.

Lattice Point Triangles

with integer side-lengths

We want lattice points $z, w \in \mathbb{R}^2$ so that $\|z\|$, $\|w\|$ and $\|z - w\|$ are integers.

Look at the plane as if it were the complex plane. Then z and w are Gaussian integers. So, we want Gaussian integers z and w so that $|z|$, $|w|$, and $|z - w|$ are integers. But if $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2} = \sqrt{N(z)}$$

Hence, to make the length an integer, make the norm a perfect square. To make the norm a perfect square, make the Gaussian integer a perfect square in $\mathbb{Z}[i]$.

Lattice Point Triangles
with integer side-lengths

That is, we want Gaussian integers z and w so that z, w, and $z - w$ are perfect squares in $\mathbb{Z}[i]$. So, we want to choose z and w so that

$$\begin{aligned} z &= \alpha^2 && \text{for some } \alpha \in \mathbb{Z}[i] \\ w &= \beta^2 && \text{for some } \beta \in \mathbb{Z}[i] \\ z - w &= \gamma^2 && \text{for some } \gamma \in \mathbb{Z}[i] \end{aligned}$$

That is, we want *Gaussian* integers α, β, and γ so that

$$\alpha^2 - \beta^2 = \gamma^2$$

or

$$\alpha^2 = \beta^2 + \gamma^2$$

Lattice Point Triangles

with integer side-lengths

The punchline: our favorite identity:

$$\left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+(2xy)^2$$

So, the trick is to pick any *Gaussian* integers x and y, and to let

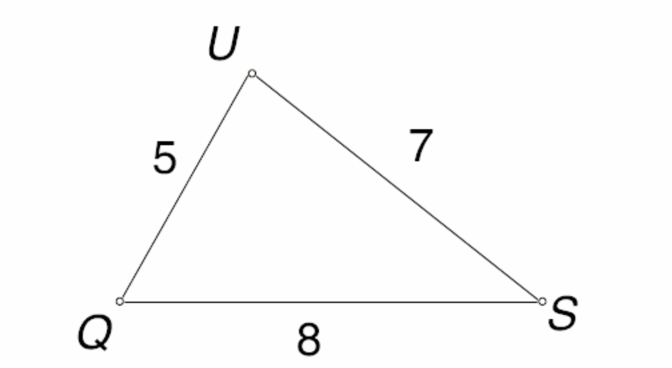
$$\alpha = x^2+y^2$$
$$\beta = x^2-y^2$$

Then let

$$z=\alpha^2$$
$$w=\beta^2$$

This method produces shows that the origin, $(-192, 256)$, and $(-60, 32)$ are vertices of an integer sided triangle. There are plenty more. See [2] for the details.

The Mathematics of Task Design



A nice triangle

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\,\cos 60^\circ \\ &= a^2 + b^2 - 2ab\,\tfrac{1}{2} \\ &= a^2 - ab + b^2 \end{aligned}$$

An Algebraic Approach

So, we want integers (a, b, c) so that

$$c^2 = a^2 - ab + b^2$$

Call such a triple an "Eisenstein Triple."
Let

$$\omega = \frac{-1 + i\sqrt{3}}{2} \quad \text{so that} \quad \omega^3 = 1$$

Note that

$$\begin{aligned} \omega + \overline{\omega} &= -1 \quad \text{and} \\ \omega\overline{\omega} &= 1 \end{aligned}$$

An Algebraic Approach

So, consider $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$.

Then $(a + b\omega)(a + b\overline{\omega}) = a^2 - ab + b^2$.

Ah. . . $a^2 - ab + b^2$ is what we want to make a perfect square. . .

An Algebraic Approach

Start with $z = 3 + 2\omega$. Square it:

$$\begin{aligned} z^2 &= (3+2\omega)^2 \\ &= 9 + 12\omega + 4\omega^2 \\ &= 9 + 12\omega + 4(-1-\omega) \quad (\omega^2 + \omega + 1 = 0) \\ &= 5 + 8\omega \end{aligned}$$

and voilà:

$$5^2 - 5 \cdot 8 + 8^2 = 49, \quad \text{a perfect square.}$$

So the triangle whose sides have length 5, 8, and 7 has a 60° angle.

An Algebraic Approach

$b \to$ a	1	2	3	4
2	$3+3\omega, 3$			
3	$8+5\omega, 7$	$5+8\omega, 7$		
4	$15+7\omega, 13$	$12+12\omega, 12$	$7+15\omega, 13$	
5	$24+9\omega, 21$	$21+16\omega, 19$	$16+21\omega, 19$	$9+24\omega, 21$
6	$35+11\omega, 31$	$32+20\omega, 28$	$27+27\omega, 27$	$20+32\omega, 28$
7	$48+13\omega, 43$	$45+24\omega, 39$	$40+33\omega, 37$	$33+40\omega, 37$
8	$63+15\omega, 57$	$60+28\omega, 52$	$55+39\omega, 49$	$48+48\omega, 48$
9	$80+17\omega, 73$	$77+32\omega, 67$	$72+45\omega, 63$	$65+56\omega, 61$
10	$99+19\omega, 91$	$96+36\omega, 84$	$91+51\omega, 79$	$84+64\omega, 76$

A Geometric Approach

A *rational point* on the unit circle leads to a Pythagorean triple:

If $\left(\frac{a}{c}, \frac{b}{c}\right)$ is on the graph of $x^2 + y^2 = 1$, then

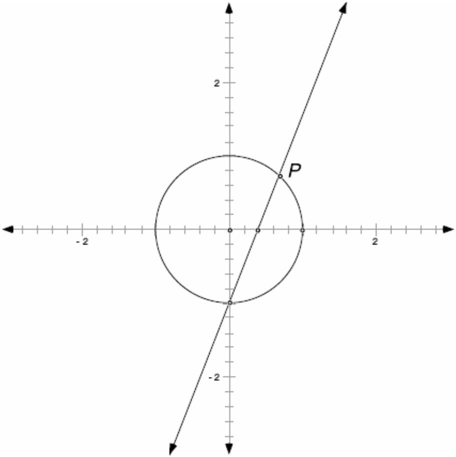
$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

or

$$a^2 + b^2 = c^2$$

So, how do you find rational points on the unit circle?

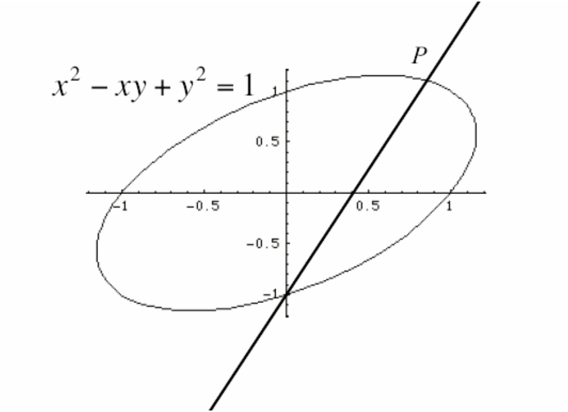
A Geometric Approach
"Sweeping Lines"



If the line has rational slope, P has rational coordinates

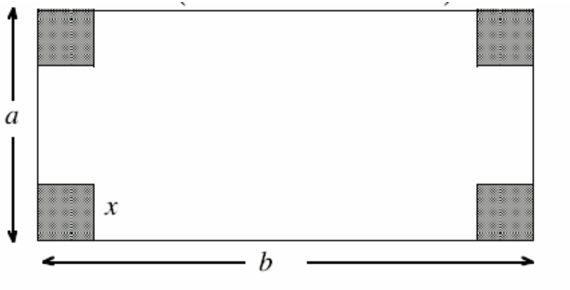
A Geometric Approach

Eisenstein triples from Rational Points



If the line has rational slope, P has rational coordinates

Other Examples
Reasonable and Rational Boxes



Soon to be a box

Well, as we tell our students, let the size of the cut-out be x. Then the volume is a function of x:

$$V(x) = (a - 2x)(b - 2x)x = 4x^3 - 2(a + b)x^2 + abx,$$

Other Examples

Reasonable and Rational Boxes

so,

$$V'(x) = 12x^2 - 4(a+b)x + ab.$$

We want this to have rational zeros, so we want the discriminant

$$16(a+b)^2 - 48ab$$

to be a perfect square. But 16 is a perfect square, so we want to make

$$(a+b)^2 - 3ab = a^2 - ab + b^2$$

a perfect square. We can do this by taking a and b to be the legs of an Eisenstein triple.

OTHER EXAMPLES

CLEAN CUBICS

We (all of us) want cubic polynomials

$$f(x) = ax^3 + bx^2 + cx + d$$

with integer coefficients, zeros, extrema, and inflection points. With a little more work, Eisenstein integers can be used here, too. The problem comes down to finding an Eisenstein integer $\alpha + \beta\omega$ so that

$$N(\alpha + \beta\omega) = 3q^2.$$

For details, see [2].

Other Examples
Clean Cubics

$s \rightarrow$ / $r \downarrow$	1	2	3
2	$54 - 27x + x^3$	$-128 - 48x + x^3$	
3	$286 - 147x + x^3$	$286 - 147x + x^3$	$-1458 - 243x + x^3$
4	$-506 - 507x + x^3$	$3456 - 432x + x^3$	$-506 - 507x + x^3$
5	$-7722 - 1323x + x^3$	$10582 - 1083x + x^3$	$10582 - 1083x + x^3$
6	$-35282 - 2883x + x^3$	$18304 - 2352x + x^3$	$39366 - 2187x + x^3$

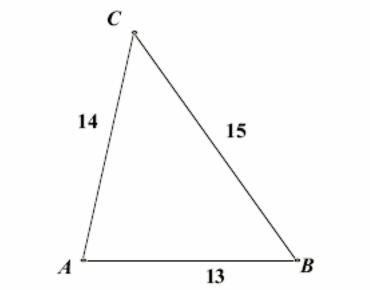
$g(x) = x^3 - 3q^2x + d$ where

$$\begin{aligned} (1+2\omega)(r+s\omega)^2 &= m + n\omega \\ 3q^2 &= N\left(m + n\omega\right) \\ d &= m\left(m^2 - 3q^2\right) \end{aligned}$$

These can be translated to produce examples with a non-zero x^2 term.

Other Examples

Heron Triangles

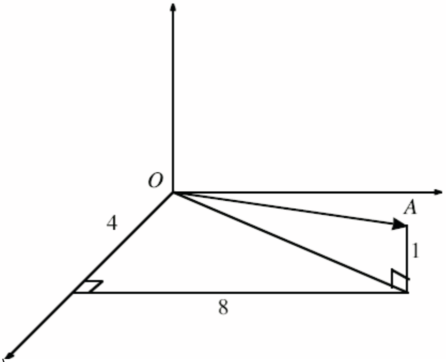


A $(13, 14, 15)$ triangle

$$\begin{aligned} s &= \tfrac{1}{2}(13 + 14 + 15) = 21 \\ A &= \sqrt{21(21-13)(21-14)(21-15)} = 84 \end{aligned}$$

Other Examples
Vectors in $\mathbb{Z}^3$

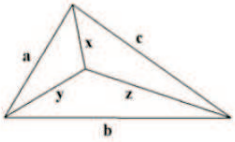
Find vectors in $\mathbb{Z}^3$ with integral length.
Ex: $(4, 8, 1)$



How long is $\overrightarrow{OA}$?

Other Examples
Nice Fermat triangles

Find *Matsuura Triples*: integer sided triangles so that the Fermat point is an integer distance from each vertex.

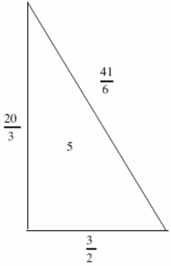


x	y	z	a	b	c
195	264	325	399	511	455
264	325	440	511	665	616
390	528	650	798	1022	910
528	650	880	1022	1330	1232
585	792	975	1197	1533	1365

Nice Fermat Triangles

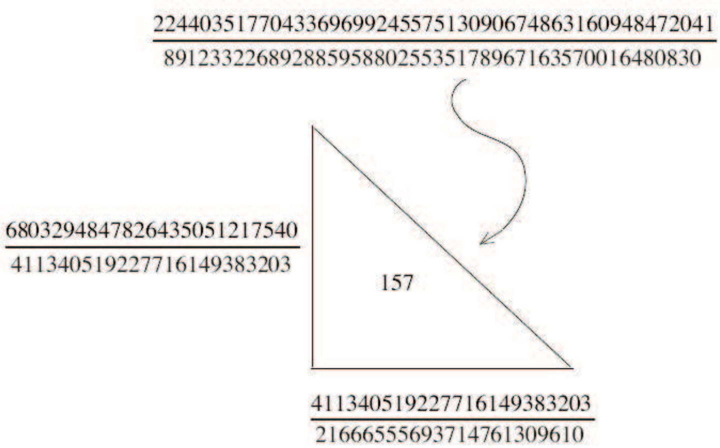
Other Examples
Congruent numbers

Find *congruent numbers*: Integers that are areas of right triangles with rational side lengths.
Example: 6 is the area of a $(3, 4, 5)$ right triangle.
Non-example: 1 is not a congruent number (Fermat).
Example: 5 is a congruent number.



Example: 157 is a congruent number.

Other Examples
Congruent Numbers



157 is a congruent number.

Outline

Nice Triangles

Find the length of the sides of the triangle whose vertices are

$$(13, -134, 80), (23, 44, 1), \text{ and } (45, 90, 0)$$

Answer: 195, 240, and 51

The Mathematics of Task Design

Take it Further

This led us to the question "Which planes in $\mathbb{Q}^3$ contain rational triangles?

A *quadratic space* is a finite dimensional vector space V over $\mathbb{Q}$, endowed with a non-degenerate symmetric bilinear form

$$B = B_V : V \times V \to \mathbb{Q}.$$

We let $q = q_V : V \to \mathbb{Q}$ be the associated quadratic form $q_V(x) = B_V(x, x)$ for all $x \in V$, and let $\delta(V)$ be the discriminant of q.

The Mathematics of Task Design

Take it Further

Theorem (Stevens) Let $V \subseteq \mathbb{Q}^3$ be a plane. Then the following are equivalent:

1. V contains a unit vector;
2. $\delta(V)$ is a sum of two squares in $\mathbb{Q}$;
3. V is isometric to an imaginary quadratic field (endowed with the norm form);
4. V contains a rational triangle.

An Example

Pick your favorite rational unit vector. Mine is $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$.

This vector belongs to lots of rational planes. For example, $2x + y - 6z = 0$.

And $2^2 + 1^2 + (-6)^2 = 41$, which equals $5^2 + 4^2$.

Outline

CONCLUSIONS

- Mathematics teaching, like any mathematical profession, involves profession-specific applications of classical mathematics.
- The mathematics involved in high school and in the teaching high of school can lead one into deep mathematical results and methods.
- The kinds of mathematics used in mathematics teaching is a valuable arena of applications for *all* mathematics majors.

References:

[1] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer Verlag, New York, 1993.

[2] A. Cuoco, “Meta-Problems in Mathematics,” *College Mathematics Journal*, November, 2000.

THANKS

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